Vulnerability Analysis of Optimized Structures

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An iterative method for the analysis of damaged structures is presented in the context of the displacement method of finite-element analysis. the damage is defined as the loss of stiffness of the structure. The method consists of first finding the solution of the undamaged structure and then determining the perturbed solution due to the damage by using the first two terms of the Taylor's series expansion of the initial solution. The elastic analysis of the effects of the damage, the elastoplastic analysis of the effects of material yielding, and the estimation of the loss of frequency and mode shape deterioration are the primary topics discussed. A brief discussion of a criterion for convergence of the solution and an empirical definition of the extent of damage is included. Applications of the procedure to an airplane wing structure and a truss are presented. This procedure is more economical, in most cases, than direct analysis of the structure with the damage or the usual piecewise linear analysis to include material nonlinearities.

Introduction

IT is a well-known fact that optimized structures are, in general, grossly nonoptimal for loads other than those for which they are designed. In particular, structures optimized for single loading conditions tend to degenerate to the most efficient determinate structures. The determinate structures have little residual strength when they are damaged. In such cases, damage to any part of the structure can produce catastrophic failures. In addition, fabrication defects and improper definition of the loading environment can impair the utility of the structure seriously. However, the situation can be improved significantly at a relatively minor cost by including more loading conditions in the optimization, even if some of them have only a minor effect on the overall strength of the structure. Design for multiple loading conditions results in redundant structures. A damaged redundant structure can redistribute loads to alternate load paths, preventing total collapse of the structure. This residual strength in structures is particularly important for vehicles operating in a combat environment. In spite of these precautions, it is impossible to protect those structures that lose large parts of the primary structure. It is possible, however, in some cases to identify critical areas and provide the necessary redundancy in the system so that a damaged vehicle at least can return safely. Such provisions in design are highly desirable for economic reasons, as well as for the safety of the crew.

A typical mission profile of a fighter aircraft includes takeoff, cruise, combat, landing, etc. This profile requires the designer to consider a larger number of design conditions for various structural components. It is assumed that the loading corresponding to each of these design conditions can be defined with reasonable certainty. However, the nature and location of the damage are impossible to predict. To study the effects of the damage, one has to examine a large number of hypothetical damage models. It would be time-consuming and expensive to modify and make a full analysis of each of these models. This paper presents an efficient prediction procedure by which the effects of damage can be studied at a relatively low cost. The prediction procedure is based on an iterative method using the first two terms of the Taylor's series expansion. This method is valid for changes in stiffness and/or mass properties of the structure. This paper examines a number of aspects of the effects of damage and establishes an empirical procedure for automated vulnerability analysis. The procedure can be used effectively to 1) identify critical areas and the corresponding loading conditions where the damage can cause catastrophic failure of the structure; 2) establish a procedure for residual strength analysis of partially damaged structures; 3) estimate the effects of redistribution of loads due to elastoplastic behavior of the structure; and 4) estimate frequency loss and mode shape degradation of partially damaged structures.

With the recent advances in analytical methods and in the efficiency of modern computers, the idea of vulnerability analysis of structures is gaining ground rapidly. The prospect of being able to optimize practical structures[1] further mandates the need for such vulnerability analysis.

Elastic Strength Estimation of Damaged Structures

It is assumed that the mathematical model of the structure is set up for analysis using the displacement method for finite-element analysis.[2] Since the proposed prediction procedure depends heavily on the original analysis of the structure, it is worthwhile reviewing the basic steps of the finite-element analysis. The first step in such an analysis is to assume appropriate shape functions in order to relate the internal displacements with a set of discrete displacement degrees of freedom of an element:

$$W_i = \phi_i \nu_i \tag{1}$$

where W_i represents the displacement field, and ν_i is a vector of the discrete generalized degrees of freedom. The shape function ϕ_i is a rectangular matrix whose elements are functions of the spatial coordinates. Using Eq. (1), the element stiffness matrices of the structure can be determined by

$$k_i = \int_{v_i} \phi_i B_i^t G_i B_i \phi_i \mathrm{d}v_i \tag{2}$$

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where B_i is a linear differential operator that expresses the strain displacement relations in the form

$$e_i = B_i w_i \tag{3}$$

and G_i, the matrix of elastic constants, relates the stresses and strains

$$\sigma_i = G_i e_i \tag{4}$$

The third step in the analysis is to determine the total structure stiffness matrix by adding the element stiffness matrices:

$$K = \Sigma a_i^t k_i a_i \tag{5}$$

where K is the structure stiffness matrix in the global coordinate system; a_i is the matrix that expresses the relation between element and structure generalized displacements by

$$v_i = a_i r \tag{6}$$

The next step in the analysis is to solve the equilibrium equations

$$Kr = R \tag{7}$$

where R is the applied force matrix. Once the displacements of the structure are determined by solution of Eq. (7), the internal displacements, stresses, and strains can be determined by Eqs. (6), (1), (3), and (4).

A large percentage of the finite-element analysis effort is involved in the solution of Eq. (7). This solution can be obtained economically by using Gaussian elimination, taking advantage of the sparseness characteristics of the stiffness matrix.[3] This procedure consists of three basic steps. The first step involves the decomposition of the original stiffness matrix by the relation

$$K = LDL^t \tag{8}$$

where L is the lower unit triangular matrix, and D is the diagonal matrix. The advantage of this decomposition scheme is that the decomposed L matrix retains some of the sparseness characteristics of the original K matrix. The next step is the forward substitution using the relation

$$LY = R \tag{9}$$

where the vector Y is given by

$$Y = DL^t r \tag{10}$$

The last step is to solve for vector r using back substitution in Eq. (10).

Of these steps, decomposition [Eq. (8)] constitutes the major effort, and the last two steps, forward and back substitutions (FBS), require very little effort. Since the decomposition is independent of the number of loading conditions, analysis for multiple loading conditions involves only the last two steps. (FBS).

Assuming that the original structure is analyzed by the foregoing procedure, the perturbed solution can be estimated by a Taylor's series expansion as follows:

$$r + \mathrm{d}r = r + \sum_{i=1}^{p} \frac{\partial}{\partial x_i}(r)\mathrm{d}x_i + \frac{1}{2!}\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}(r)\mathrm{d}x_i^2 + \ldots \tag{11}$$

where x_i is an implicit parameter whose change will affect the stiffness of the structure. It is assumed in Eq. (11) that p such parameters have changed in the structure. Taking only the first two terms of the Taylor's series, the differential change in response can be written as

$$\mathrm{d}r = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}(r)\mathrm{d}x_i + \frac{1}{2!}\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}(r)\mathrm{d}x_i^2 \tag{12}$$

Multiplying both sides of Eq. (12) by the original stiffness matrix gives

$$K\mathrm{d}r = K\sum_{i=1}^{p} \frac{\partial}{\partial x_i}(r)\mathrm{d}x_i + \frac{1}{2!}K\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}(r)\mathrm{d}x_i^2 \tag{13}$$

Using the original equilibrium equation [Eq. (7)], a further approximation of Eq. (13) can be written as

$$K\mathrm{d}r = -\mathrm{d}K[r + \mathrm{d}r] \tag{14}$$

where $\mathrm{d}K$ in Eq. (14) is given by

$$\mathrm{d}K = \sum_{i=1}^{p} \mathrm{d}K_i = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}(K)\mathrm{d}x_i \tag{15}$$

From Eq. (14), the iterative algorithm for the perturbed solution can be written as

$$K\mathrm{d}r^{(\nu+1)} = -\mathrm{d}K[r + \mathrm{d}r^{(\nu)}] \tag{16}$$

where ν stands for the cycle of iteration. A comparison of Eqs. (7) and (16) reveals some interesting facts. The object of the solution of Eq. (7) is to obtain the unknown displacement vector r for a known load vector R, whereas in Eq. (16) the object is to obtain the perturbed $\mathrm{d}r^{(\nu+1)}$ for the known right-hand side from the previous iteration. In both equations, however, the stiffness matrix K is the same. Since this stiffness matrix already is decomposed [using Eq. (8)] for the solution of Eq. (7), it is readily available for the solution of Eq. (16). Also, it need not be redetermined in each iteration. Only the forward and back substitution steps have to be repeated.

Similar procedures for approximate reanalysis (in static case) using an equivalent of the first two terms of Taylor's series were discussed.[4-6] However, in all of the previous cases the iteration was extended over the entire displacement vector instead of only the differential displacement vector, as in the present case. The latter procedure assures a more stable convergence. The first-order approximation (only one term of the Taylor's series) gives an explicit expression for $\mathrm{d}r$ without iteration.[3,7-9] However, the approximation is inadequate in case of large changes in the response.

Before discussing the convergence criteria, the definition of certain terms is required. The displacement vector r is the solution of the original undamaged structure. The vector $\mathrm{d}r$ is the true perturbed solution of the damaged structure. The actual response of the damaged structure is r^D, and it can be obtained either by the relation

$$r^D = r + \mathrm{d}r \tag{17}$$

or by the solution of the equilibrium equations of the damaged structures, which are given by

$$K^D r^D = R \tag{18}$$

where K^D is the acutal stiffness matrix of the damaged structure. The applied load matrix R is assumed to be the same for both damaged and undamaged models. The perturbed solution can be written as

$$\mathrm{d}r = \lim \mathrm{d}r^{(\nu)}; \quad \nu \rightarrow 1,2,\ldots,\infty \tag{19}$$

where $\mathrm{d}r^{(\nu)'}$ is obtained from Eq. (16). Now the question is, can it be established that Eq. (16) guarantees a reasonably rapid rate of convergence? The following argument will provide some understanding of the nature of the convergence. Equation (16) can be written as

$$\mathrm{d}r^{(\nu+1)} = -K^{-1}\mathrm{d}K[r + \mathrm{d}r^{(\nu)}] \tag{20}$$

This equation also can be written as

$$\mathrm{d}\boldsymbol{r}^{(\nu+1)} = [\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2+\ldots+\boldsymbol{A}^\nu]\boldsymbol{A}\boldsymbol{r} \tag{21}$$

where $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = -\boldsymbol{K}^{-1}\mathrm{d}\boldsymbol{K} \tag{22}$$

An iteration using Eq. (21) converges only if $\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2+\boldsymbol{A}^3+\ldots+\boldsymbol{A}^\nu$ converges.[10] This convergence depends on the nature of the matrix $\boldsymbol{A}$. If $\boldsymbol{A}^\nu \to 0$ as ν increases, then convergence is assured. The latter condition is assured for small changes in the stiffness of the structure.[11]

Of course a mere assurance of convergence is not enough justification for the use of this method. This prediction has to be competitive with the direct analysis of the modified model of the damaged structure. This depends on the number of cycles of iteration [Eq. (16)] that are needed for the solution. The rate of convergence depends on the effect of the damage on the response of the structure. A quantitative understanding of this effect can be obtained by comparing the strain energies (or the work done by the external forces) of the damaged and the undamaged structures. Assuming that the damage is not severe enough to produce collapse of the structure, the work equation for the original and the damaged structures can be written as

$$W = \tfrac{1}{2}\boldsymbol{R}^t\boldsymbol{r} \tag{23}$$

$$W^D = \tfrac{1}{2}\boldsymbol{R}^t\boldsymbol{r}^D \tag{24}$$

If the difference between W and W^D is within 10 to 20%, the damage will be considered as minor; between 20 and 50% as medium; and between 50 and 100% as major. Over 100% is considered as producing collapse of the structure. It should be realized that these definitions are arbitrary, but some such classification is necessary to establish the limitations of the iterative method. A reasonable estimate, based on experience in using the program developed for this effort, is that about 30 cycles of iteration would be equivalent to the time required for a direct analysis of the damaged structure. In both cases, it is assumed that the initial solution of the undamaged structure is available.

An approximate breakdown of the number of cycles required for each damage range is as follows: minor, 3-5 cycles; medium, 5-15 cycles; and major, 15-30 cycles. This means that, in case of minor damage, 5-10 damage cases can be investigated for the same cost as a direct analysis. Similarly, 2-5 damage cases for medium and 1-2 cases for major damage can be studied for the same analysis cost. There is a good possibility that the convergence rate can be improved further, but this aspect will be deferred for future investigation.

Since W^D is not known in advance, it cannot be used for convergence criterion. To establish such a criterion it is necessary to examine the work of the external forces and the internal strain energy defined as

$$W^{(\nu)} = W + \mathrm{d}W^{(\nu)} \tag{25}$$

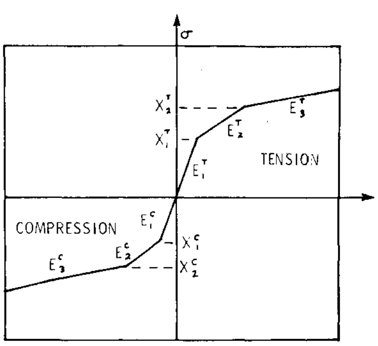

Fig. 1 Typical stress-strain diagram.

$$U^{(\nu)} = \frac{1}{2}\sum_{i=1}^{n}\int_{v_i}\boldsymbol{\sigma}_i^{(\nu)}\boldsymbol{e}_i^{(\nu)}\,\mathrm{d}v_i \tag{26}$$

where $\mathrm{d}W^{(\nu)}$ is given by

$$\mathrm{d}W^{(\nu)} = \tfrac{1}{2}\boldsymbol{R}^t\mathrm{d}\boldsymbol{r}^{(\nu)} \tag{27}$$

and $\boldsymbol{\sigma}^{(\nu)}$ and $\boldsymbol{e}^{(\nu)}$ are the stress and strain vectors corresponding to the displacement vector $\boldsymbol{r}+\mathrm{d}\boldsymbol{r}^{(\nu)}$. If the system is in equilibrium, which respresents the converged solution, the following relation must be satisfied:

$$W^{(\nu)} = U^{(\nu)} \tag{28}$$

The difference between these two quantities can be used as a criterion for convergence. Determination of $W^{(\nu)}$ is quite inexpensive, since it involves only the scalar product of two vectors. However, computation of $U^{(\nu)}$ in each cycle can be an expensive proposition, since it involves computation of stresses and strains in all of the elements. In practice, however, it would be adequate to specify a reasonable number of cycles and check the rate of change in $\mathrm{d}W^{(\nu)}$ which can be used as an approximate criterion for convergence. While obtaining the stress information at the end, however, the strain energy $U^{(\nu)}$ of the structure can be determined and compared with $W^{(\nu)}$. If the difference between these two quantities is over the prescribed bounds, the iteration can be continued. The results of the application of the procedure are presented in the final section of the paper.

Estimation of the Effects of Yielding

The piecewise linear analysis is the most commonly accepted procedure for introducing the effects of material nonlinearities into the finite-element analysis of structures.[12] This procedure consists of dividing the applied load into a number of small load steps with an updated stiffness matrix to account for the variation of the modulus of elasticity with the stress level. The modulus of elasticity in each step is determined from the stress level in each member in the previous step. The solution obtained in each step is added to the previous solution, and the analysis of the next step is carried out until the full load is reached. This procedure involves the solution of Eq. (7) as many times as there are steps. Normally a reasonable solution requires a minimum of 6 to 8 such steps. In the case of large structures, 6 or 8 analyses can be prohibitively expensive. The cost of elastoplastic analysis can be reduced considerably by adopting the procedure described in the previous section. The details of the extension of this procedure are described in the remainder of this section.

The first step in this procedure is to make a linear analysis of the structure for the full load using Eq. (7). The next step is to determine the effective stress ratios for each element. These ratios are determined by appropriate failure criterion. One such criterion is the modified Von Mises or energy of distortion criterion adopted here for membrane elements:

$$\theta_i = \sqrt{\left(\frac{\sigma_x}{x_l}\right)_i^2 + \left(\frac{\sigma_y}{y_l}\right)_i^2 - \left(\frac{\sigma_x\sigma_y}{x_l y_l}\right)_i + \left(\frac{\sigma_{xy}}{z_l}\right)_i^2} \tag{29}$$

where σ_x, σ_y, and σ_{xy} are the actual stresses in the elements. x_l, y_l, and z_l are the respective stresses corresponding to the first break in the stress-strain diagram of the material (Fig. 1). The stresses x_l, y_l, and z_l can be different in tension and compression. θ_i is the effective stress-rate for the ith element.

Assuming that there is a single break in the stress-strain diagram, the elements can be divided into elastic and inelastic groups. The elements with $\theta \leq 1$ belong to the elastic group, and those with $\theta > 1$ belong to the inelastic group. From the effective stress ratio θ, two load factors are defined as

$$\alpha = 1/\theta_{\max} \tag{30}$$

$$\beta = (1-\alpha) \tag{31}$$

where θ_{max} is the maximum effective stress-ratio of all of the elements. Using these factors, the total applied load is divided into two parts, $\alpha \boldsymbol{R}$ and $\beta \boldsymbol{R}$. The corresponding displacement vectors are $\boldsymbol{r}^\alpha$ and $\boldsymbol{r}^\beta$, and they are obtained simply by scaling the displacement determined in the linear analysis:

$$\boldsymbol{r}^\alpha = \alpha \boldsymbol{r} \tag{32}$$

$$\boldsymbol{r}^\beta = \beta \boldsymbol{r} \tag{33}$$

where $\boldsymbol{r}$ is the displacement vector obtained in the first step. Now the total response can be divided into two parts. The linear response is given by $\boldsymbol{r}^\alpha$, and the elastoplastic response can be determined by prediction [Eq. (16)] as follows:

$$\boldsymbol{K}\mathrm{d}\boldsymbol{r}^{(\nu+1)} = -\mathrm{d}\boldsymbol{K}[\boldsymbol{r}^\beta + \mathrm{d}\boldsymbol{r}^{(\nu)}] \tag{34}$$

where $\boldsymbol{K}$ is the original linear elastic stiffness matrix, and the differential stiffness matrix $\mathrm{d}\boldsymbol{K}$ is given by

$$\mathrm{d}\boldsymbol{K} = \sum_{i=p+1}^{n} \eta_i \mathrm{d}\boldsymbol{K}_i \tag{35}$$

where $\mathrm{d}\boldsymbol{K}_i$ are the initial elastic stiffness matrices of the elements. The stiffness reduction factors η_i are determined by

$$\eta_i = (E_1^i - E_2^i)/E_1^i \tag{36}$$

where E_1^i and E_2^i are the modulus of the material of the ith element before and after the first break in its stress-strain diagram. It should be noted that the summation in Eq. (35) extends only over the inelastic elements.

The solution of Eq. (34) requires the decomposition of the stiffness matrix $\boldsymbol{K}$, which is readily available from the linear analysis in the first step. Only the forward and back substitutions have to be repeated in each cycle of iteration. The total elastoplastic response $\boldsymbol{r}^e$ of the material is given by

$$\boldsymbol{r}^e = \boldsymbol{r}^\alpha + \mathrm{d}\boldsymbol{r} \tag{37}$$

$$\boldsymbol{\sigma}_i^e = \boldsymbol{\sigma}_i^\alpha + \mathrm{d}\boldsymbol{\sigma}_i \tag{38}$$

where $\mathrm{d}\boldsymbol{r}$ is the displacement vector obtained from Eq. (34), and $\mathrm{d}\boldsymbol{\sigma}_i$ is the stress vector in the ith element due to the displacement vector $\mathrm{d}\boldsymbol{r}$. This stress vector is determined by using E_1 for the elastic elements and E_2 for the inelastic elements. The stress vector $\boldsymbol{\sigma}^\alpha$ corresponds to the linear response $\boldsymbol{r}^\alpha$ for all of the elements. In all of these computations, it is assumed that the Poisson ratio does not change during yielding.

Extension of this procedure to the case of multiple breaks in the stress-strain diagram is straightforward. However, the procedure has to be repreated as the new breaks are encountered. A brief outline of this extension is given here, and the details and possible improvements are discussed in Ref. 6. As described previously, the first step is to make a linear analysis of the structure for the full load. The next step is to determine the effective stress ratio vectors $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)} \ldots \boldsymbol{\theta}^{(q)}$. The vector $\boldsymbol{\theta}^{(i)}$ corresponds to the ith break in the material stress-strain diagram. The effective stress ratio is given by Eq. (29) and is modified for the multiple breaks:

$$\theta_i^{(j)} = \sqrt{\left(\frac{\sigma_x}{x_j}\right)_i^2 + \left(\frac{\sigma_y}{y_j}\right)_i^2 - \left(\frac{\sigma_x \sigma_y}{x_j y_j}\right)_i + \left(\frac{\sigma_{xy}}{z_j}\right)_i^2} \tag{39}$$

It should be noted that the numerators of the expression under the squre root are the same as before, but x_j, y_j, and z_j are the stress levels corresponding to the ith break. The next step is to determine the ratios $\alpha^{(1)}, \alpha^{(2)} \ldots \alpha^{(q)}$ and $\beta^{(1)}, \beta^{(2)} \ldots \beta^{(q)}$. These ratios are given by

$$\alpha^{(i)} = 1/\theta_{max}^{(i)} \tag{40}$$

$$\beta^{(i)} = \alpha^{(i+1)} - \alpha^{(i)} \tag{41}$$

With these definitions, the iteration can be carried out at each step by using Eq. (34). The total response can be written as

$$\boldsymbol{r}^e = \boldsymbol{r}^{\alpha^1} + \mathrm{d}\boldsymbol{r}^{\beta^1} + \ldots + \mathrm{d}\boldsymbol{r}^{\beta^\nu} \tag{42}$$

It should be noted that, in computing the differential stiffness in each step, the stress levels obtained by using the displacement vector computed in the last step are used. It is very important to note that no new analysis is required in solving Eq. (34). The original decomposed stiffness matrix can be used at all of the steps; only the forward and back substitutions have to be repeated. Because of this feature, the elastoplastic analysis requires only one linear analysis and a fraction of the effort required of a new analysis. To obtain equal accuracy by a piecewise linear analysis, at least six to eight or more load increments are necessary. Each load increment normally would require a new analysis. The final section of the paper contains the results of an elastoplastic analysis using this prediction procedure. These results also are compared with those obtained by a piecewise linear analysis using NASTRAN.[12]

Estimation of Frequency and Mode Shape Degradation in Damaged Structures

In assessing the frequency and mode shape degradation, only the elastic case will be considered here. The loss of frequency due to damage can be interpreted as a measure of the stiffness deterioration. However, it is more difficult to give mode shape deterioration a quantitative interpretation unless some sort of dynamic response analysis is made. The present investigation considers only the effects of damage on free vibration, and the response analysis will be deferred to later investigation.

The equilibrium equations for free vibration are given by

$$\boldsymbol{M}\ddot{\boldsymbol{r}} + \boldsymbol{K}\boldsymbol{r} = 0 \tag{43}$$

where the stiffness matrix $\boldsymbol{K}$ is the same as that given by Eq. (5). The mass matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \boldsymbol{M}_s + \boldsymbol{M}_n \tag{44}$$

where $\boldsymbol{M}_s$ and $\boldsymbol{M}_n$ are the structural and nonstructural masses, respectively. The structural mass in finite-element analysis can be written as

$$\boldsymbol{M}_s = \sum_{i=1}^{n} \boldsymbol{a}_i^t \boldsymbol{m}_i \boldsymbol{a}_i \tag{45}$$

where $\boldsymbol{m}_i$ is the element structural mass.

The solution of Eq. (43) reduces to a standard eigenvalue problem

$$\lambda \boldsymbol{M}\boldsymbol{\psi} = \boldsymbol{K}\boldsymbol{\psi} \tag{46}$$

where λ represents the square of the natural frequency of the structure, and ψ is the eigenvector describing the corresponding mode shape. By premultiplying both sides of Eq. (46) by ψ^t, the expression for λ is given by the Rayleigh quotient:

$$\lambda = \psi^t \boldsymbol{K}\psi / \psi^t \boldsymbol{M}\psi \tag{47}$$

From Eq. (47), the change in natural frequency due to damage can be obtained by a Taylor's series expansion. The first term of the Taylor's series gives the expression for dλ as follows[13]:

$$\mathrm{d}\lambda = \frac{\psi' \mathrm{d}\boldsymbol{K}\psi - \lambda\psi' \mathrm{d}\boldsymbol{M}\psi}{\psi' \boldsymbol{M}\psi} \tag{48}$$

where d$\boldsymbol{K}$ and d$\boldsymbol{M}$ represent the change in stiffness and mass matrices due to damage. If the eigenvalues and eigenvectors of Eq. (46) are known already, determination of dλ by Eq. (48) is straightforward and requires no iteration. However, since the damage can affect the response rather drastically, two terms of the Taylor's series will be used for better predictions of the change in frequencies. In doing so, however, the second term is simplified by assuming that the first-order approximation (total differential) is valid for both the eigenvalues and eigenvector. This assumption leads to an approximation for dλ as given by[11]

$$\mathrm{d}\lambda = \frac{\psi' \boldsymbol{B}(\psi + \mathrm{d}\psi)}{(1+C)} \tag{49}$$

where the matrix $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \frac{\mathrm{d}\boldsymbol{K} - \lambda \mathrm{d}\boldsymbol{M}}{\psi' \boldsymbol{M}\psi} \tag{50}$$

and C is given by

$$C = \frac{\psi' \mathrm{d}\boldsymbol{M}\psi + \psi' \boldsymbol{M}\mathrm{d}\psi}{\psi' \boldsymbol{M}\psi} \tag{51}$$

If the changes in mode shape and the modal mass matrix are assumed to be negligible, then Eq. (49) reduces to Eq. (48).

However, to determine dλ from Eq. (49), the change in mode shape dψ must be known. The approximate expression for dψ can be determined by taking again the first two terms of the Taylor's series and using Eq. (46). This equation has to be an iterative equation for solution, and it is given by[11]

$$[\boldsymbol{K} - \lambda\boldsymbol{M}]\mathrm{d}\psi^{(\nu+1)} = \boldsymbol{D}[\psi + \mathrm{d}\psi_n^{(\nu)}] \tag{52}$$

where the matrix $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = -\mathrm{d}\boldsymbol{K} + \lambda\mathrm{d}\boldsymbol{M} + \mathrm{d}\lambda\boldsymbol{M} + \mathrm{d}\lambda\mathrm{d}\boldsymbol{M} \tag{53}$$

By iteration using Eqs. (49) and (52), the changes in frequency and mode shape can be determined.

The outline of the iterative procedure is as follows:

1) First the value of dλ is determined by using Eq. (48) instead of Eq. (49). This is equivalent to the assumption that the change in mode shape is small and can be neglected.

2) The next step is to determine dψ using Eq. (52), with the initial dψ assumed to be a null matrix. If the original eigenvectors are determined by inverse iteration,[13,14] then the decomposed dynamic matrix $(\boldsymbol{K} - \lambda\boldsymbol{M})$ is readily available, and only the forward and back substitutions have to be repeated for the solution of Eq. (52). Otherwise, decomposition of $(\boldsymbol{K} - \lambda\boldsymbol{M})$ can be done once at the beginning of the iteration.

3) The dψ determined in the second step can be used in Eq. (49) to correct the value of dλ.

4) The last step is to repeat step 2 with the dλ computed in step 3. Further details of this procedure and the results of its application are given in Ref. 11.

It is interesting to note that the derivatives of the response quantities such as eigenvalues, eigenvectors, and the flutter

Table 1 Description of cases

Structure	Case	Loading	Description of design	Damage description	Weight, lb Undamaged	Weight, lb Damaged
10-bar truss	1	1	Optimized structure	Member 9 removed	5062	3988
	2	1	Optimized structure	Member 9 reduced to 1/10	5062	4096
	3	1, 2	Optimized structure	Member 9 removed	5594	4664
	4	1, 2	Optimized structure	Member 9 reduced to 1/10	5594	4757
	5	1	Optimized structure	Members 2, 5, 6, 10 removed	5062	5031
	6	1	Equal sizes	Member 9 removed	8266	7263
	7	1	Equal sizes	Member 9 reduced to 1/10	8266	7364
	8	1	Equal sizes	Members 2, 5, 6, 10 removed	8266	5136
	9	1	Same as case 2	Elastoplastic analysis	4096	4096
	10	1	Same as case 9	Piecewise linear analysis (using NASTRAN level 16)	4096	4096
Wing	11	1	Optimized structure	Members 6, 7, 26, 27, 47 removed	303	255
	12	1	Same as case 11 except members 6, 7 26, 27, 47 reduced to half their sizes	Elastoplastic analysis	279	279

Table 2 Results: truss and wing analysis[a]

	Initial structure		Damaged structure								
			Analysis		Prediction						Damage
Case	Strain energy	Max. disp.	Strain energy	Max. disp.	Strain energy	Work	% diff.	Max. disp.	Cycles	range
1	0.1809	−2.00	7.5660	−148.27	7.5460	7.5560	0.13	−148.06	2000	Collapse
2	0.1809	−2.00	0.3925	−6.18	0.3924	0.3925	0.02	−6.18	100	Collapse
3	0.1534	−2.00	0.4096	−6.39	0.4096	0.4096	0	−6.39	120	Collapse
4	0.1534	−2.00	0.2660	−3.93	0.2660	0.2660	0	−3.93	50	Major
5	0.1809	−2.00	0.1821	−1.98	0.1821	0.1821	0	−1.98	4	Minor
6	0.1457	−2.00	0.1770	−2.58	0.1769	0.1770	0.04	−2.58	20	Medium
7	0.1457	−2.00	0.1685	−2.42	0.1685	0.1685	0	−2.42	20	Medium
8	0.1457	−2.00	0.1598	−2.31	0.1596	0.1596	0	−2.30	10	Minor
9	0.3925[b]	−6.18[b]	Elastoplastic analysis					−17.55	20	...
10			NASTRAN piecewise linear analysis (23 load increment)					−17.30	...	...
11	0.0928	5.83	0.1805	9.31	0.1801	0.1803	0.11	9.31	100	Major
12	0.1052[b]	6.42[b]	Elastoplastic analysis					38.67	20	...

[a] Note: Multiply strain energy by 10^6. All displacements are in inches. [b] Elastic solution.

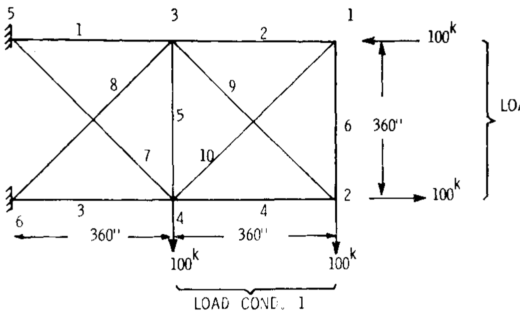

Fig. 2 a) Tenn-bar-truss. Note: The two load conditions do not act at the same time.

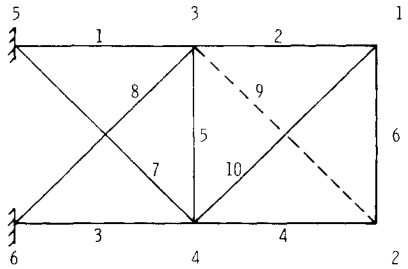

Fig. 2 b) Truss with damaged member (dotted lines).

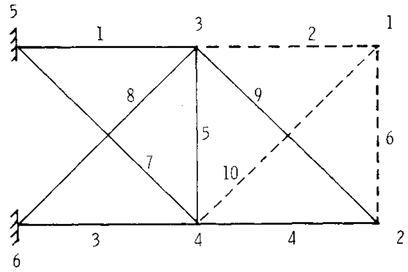

Fig. 2 c) Truss with damaged member (dotted lines).

velocity are used frequently in finding the directions of travel in many optimization schemes.[15-17] Most of these derivatives are computed by using an equivalent of the first term of the Taylor's series. However, for large changes in response, inclusion of the second terms can improve the rate of convergence to the optimum solution.

Results and Conclusions

The prediction algorithms developed in the last three sections have many common features. In all three cases, the decomposed stiffness or dynamic maxtrix is assumed to be available from the initial soltuion of the problem. The perturbed solution is obtained simply by forward and back substitution with the pseudoload vectors. In addition, the differential stiffness and mass matrices (dK and dM) are sparsely populated. This fact can be used to an advantage in repeated multiplication during an iteration. The elastic strength estimation procedure and the elastoplastic analysis procedure can be combined to predict the behavior of the damaged structure up to failure. However, for proper understanding of their independent behavior and verification, the two procedures were applied separately to the examples discussed in this section. The procedures were tested on two structures, each having a number of subcases.

The first structure is a 10 bar truss that has been discussed frequently in structural optimization literature in recent years.[1,18-20] A schematic of the structure and its loading conditions are shown in Fig. 2a. The damaged elements are shown in Figs. 2b and 2c. Table 1 lists the various cases

COORDINATES			
Node	X	Y	Z
1	0.0	0.0	2.250
5	24.0	0.0	3.000
9	48.0	0.0	2.250
51	45.0	90.0	1.125
55	60.0	90.0	1.500
59	75.0	90.0	1.125

Note: Wing is Symmetric About its Midplane

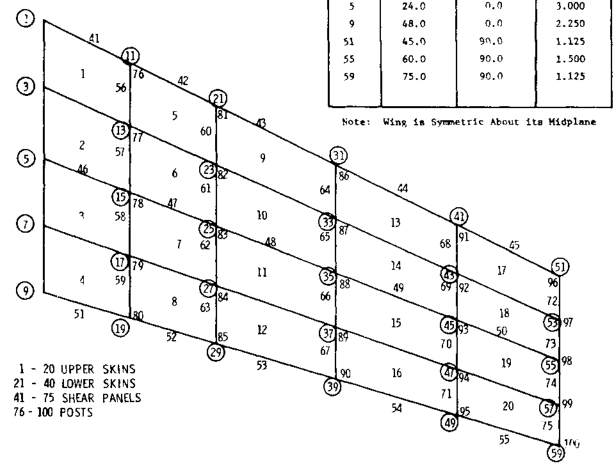

Fig. 3 Wing.

considered for this problem. In all cases, the original structure was designed with stress and displacement constraints. The maximum stress in the members was limited to 25,000 psi, and the maximum vertical displacement was limited to 2 in. at all of the modes. These limits apply only to the undamaged structure. The material of the structure was assumed to be aluminum with a modulus of elasticity $E = 10.0 \times 10^6$ psi and a density $\rho = 0.1$ lb/in.^{3}. With these design conditions, eight damage cases were considered. The first five cases used the optimized sizes. (The details of the optimization procedure are given in Ref. 1.) The next three cases used equal-size members. In all eight cases, the initial designs has the same stress and displacement limits. The last two cases were for studying the effect of material yielding. For these cases, the member sizes were the same as in case 2. The members were assumed to yield when the stresses exceed 25,000 psi. A stress-strain diagram with a single break was used in this analysis. The modulus beyond the break was assumed to be 2.0×10^6 psi. Yielding was considered only in cases 9 and 10. All other cases were elastic solutions.

In this first case, removal of member 9 increases the vertical displacement of node 2 from 2 in. to over 148 in. This clearly represents a collapse condition many times over. Even in this severe case, iteration using Eq. (16) converges to the exact solution but required over 2200 cycles. As indicated, earlier the optimization for the single loading reduced the structure to the most efficient determinate structure. Removal of a key member has essentially introduced kinematic instability into the structure. In the second case, retention of 10% of the member area reduced the severity, but the solution still represents a collapse condition. In the third case, the structure was designed for two loading conditions, and the resulting increase in weight of the structure is only 12%. However, this structure is not as vulnerable on removal of member 9 as the one designed for single loading. In the fifth case, removal of members 2, 5, 6, and 10 (even in case of optimized structure) did not affect the response as severely. The nonoptimal structure in the sixth case also was less vulnerable for damage.

The last two cases represent elastoplastic analysis. In both cases, member 9 was reduced to 1/10 of its optimized size, as in case 2. The prediction procedure described previously gave a tip displacement of 17.55 in. (node 2) in 20 cycles. In the tenth case, the elastoplastic analysis was made by piecewise linear analysis using NASTRAN[12] (level 16, rigid format 6). With six load increments, NASTRAN predicted 16.07 in. for the tip displacment. When the number of load increments was increased to 9, 15, and 23, the predicted tip displacements also increased to 17.18, 17.28, and 17.30 in., respectively. Obviously piecewise linear analysis with such a large number of load increments would be prohibitively expensive in the case of large structures. The prediction procedure presented in this paper is much more economical in such cases.

The second structure is a typical wing of an aircraft and is shown in Fig. 3. The structure is idealized by bars, membrane quadrilaterals, and shear panels. The top and bottom skins are represented by membrane quadrilaterals, spars and ribs by shear panels, and posts connecting top and bottom nodes by bars (axial force members). The quadrilaterals and shear panels were constructed out of four membrane triangles with a fictitious node at the center. In case of shear panels, only the shear strain energy was assumed to contribute to its stiffness. The material was assumed to be aluminum with the properties $E=10.5\times10^6$ psi, $\nu=0.25$, $\rho=0.1$ lb/in.^{3}. The loading on the wing was computed by assuming a 6-g symmetrical maneuver of a 40,000-lb aircraft. The spanwise distribution of the airload is assumed to be uniform in two segments. The first segment consists of the first three bays from the root, and the second segment covers the remaining two bays. The load intensity varies in a 2:1 ratio between the two segments. Chordwise, the load distribution varies in a 3:2:1 ratio. The first figure covers two panels in from the leading edge; the second and third figures cover the remaining panels. The wing was optimized for stress constraints only. The normal and shear stresses were limited to 30,000 and 18,000 psi, respectively. The energy of distortion criteria [Eq. (29)] was used in determining the effective stress. Table 1 gives the details of the damage, and Table 2 gives the results of the direct analysis and the prediction procedure.

Elastoplastic analysis of the same structure was made with the members 6, 7, 26, 27, and 47 reduced to half their optimized sizes. A stress-strain diagram with a single break was used in this analysis. The modulus of the material was assumed to change at an effective stress value of 30,000 psi. This limiting stress value was applied for shear panels also. The modulus beyond the break was assumed to be 2.0×10^6 psi. The results of this analysis could not be compared with those of the piecewise linear analysis using NASTRAN, because there is no provision for including shear panels in the NASTRAN analysis. Some additional results and the details of all of the cases considered here are given in Ref. 11. The prediction procedure for damage analysis presented in this paper is a simple and natural extension of a very effective and popular finite-element analysis method.

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